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Verification of a Conjecture of E. Thomas

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We prove that the equation $X^3-(n-1)X^2Y-(n+2)XY^2-Y^3=\pm 1$, where $n \in \mathbb{N}$, has only trivial solutions for $n>3$. © 1993 Academic Press, Inc.

INTRODUCTION

Recently E. Thomas [T] studied the following family of cubic equations

$$X^3-(n-1)X^2Y-(n+2)XY^2-Y^3=\pm 1, \qquad n \in \mathbb{N}. \tag{$*$}$$

For all n this equation has the three solutions $\pm(0, 1)$, $\pm(1, 0)$, $\pm(1, -1)$; which he called the *trivial* solutions. Thomas proved the remarkable result that for $n \geqslant 1.365 \times 10^7$ this equation has only trivial solutions. Moreover, he also proved that for $n \leqslant 10^3$ non-trivial solutions exist only for $n=0$, 1, and 3.

He conjectured that, except for these three cases, there are never non-trivial solutions. The aim of the present paper is to verify this conjecture.

1. FROM THE FAMILY OF CUBICS TO A FAMILY OF LINEAR FORMS IN LOGARITHMS

The facts quoted in this section are extracted from E. Thomas' paper [T]. We keep his notations. The equations $F_n(x)=x^3-(n-1)x^2-(n+2)x-1=0$ have three real roots

$$\lambda=\lambda(n)=\lambda_0=\lambda_{0,n}>\lambda_1=\lambda_{1,n}>\lambda_2=\lambda_{2,n}.$$

This roots verify $\lambda_1=-1/(\lambda+1)$, $\lambda_2=-(1+1/\lambda)$, and $n<\lambda<n+1$.

If (p, q) is a non-trivial solution of $(*)$, set $\gamma_i=p-q\lambda_i$, $i=0, 1, 2$. There are positive rational integers $A=A_n$ and $B=B_n$ such that

$$\gamma_i=\sigma^i(\lambda^{-A}\lambda_1^B), \qquad i=0, 1, 2,$$

where σ is the $\mathbb{Q}$-isomorphism which sends λ_i on $\lambda_{(i+1) \bmod 3}$.

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Put $k_0 = k_0(n, q) = B(\mathrm{Log}\,\lambda)/(\mathrm{Log}\,q)$. Then

$$k_0(n, q) < \left(1 + \frac{\mathrm{Log}(\lambda+2)}{\mathrm{Log}\,q}\right) N_\lambda, \qquad \text{where} \quad N_\lambda = 1 + \frac{1}{\lambda\,\mathrm{Log}(\lambda+1)}.$$

Set $\delta_0 = \delta_{0,n} = \lambda_2/\lambda_1$, $\delta_1 = \lambda/\lambda_2$, $\delta_2 = (\lambda - \lambda_1)/(\lambda - \lambda_2)$, $\delta_3 = -1/\delta_2$, and

$$\Lambda' = \Lambda'_n = A_n \,\mathrm{Log}\,|\delta_{0,n}| - B_n \,\mathrm{Log}\,|\delta_{1,n}| + \mathrm{Log}\,|\delta_{3,n}|.$$

Then

$$0 < |\Lambda'| < 1/(q^3 \lambda^4).$$

The linear form Λ'_n can also be written

$$\Lambda'_n = C_n \,\mathrm{Log}(\lambda+1) - D_n \,\mathrm{Log}\,\lambda = C_n \,\mathrm{Log}(1 + 1/\lambda) - E_n \,\mathrm{Log}\,\lambda,$$

where

$$\lambda = \lambda(n), \qquad C = 2A + B + 1, \qquad D = 2B + A + 1,$$
$$E = D - C = B - A > 0.$$

Remark that g.c.d. $(C, D) \leqslant E = D - C$. Moreover

$$D \geqslant E\lambda \,\mathrm{Log}\,\lambda.$$

This inequality replaces inequality (16) of [T]; it is implicitly proved in [T] and will be very useful here. And we have also

$$q \geqslant \lambda^{D/3k_0}.$$

Using the main result of [MW1], Thomas proved that

$$D < 27621(\mathrm{Log}\,\lambda)(3 + \mathrm{Log}\,D)^2.$$

Then, using some of the previous estimates, he got $n < 1.365 \times 10^7$.

Here, we shall first refine the lower bounds of the special linear forms Λ'_n to improve this result. Notice that results of [MW2], which are sharper than the corresponding ones proved in [MW1], lead directly to $n < 9 \times 10^6$.

2. A Lower Bound for the Linear Forms of This Family

To get a sharp result, we study the present case in detail. We do not apply a general result: we use Theorem 5.11 of [MW2]. We consider

$$\Lambda = -\Lambda'_n/C_n = (E/C)\,\mathrm{Log}\,\lambda - \mathrm{Log}(1 + 1/\lambda).$$

With the notations of [MW2] : $\alpha_1 = \lambda$, $\alpha_2 = 1 + 1/\lambda$, $d = 3$ (here d is the degree), the logs are the principal determinations of the logarithm of α_j, and we put $l_j = |\text{Log}\, \alpha_j|$, for $j = 1, 2$.

Moreover $\beta = E/C = b_1/b_2$, where $(b_1, b_2) = 1$, $b_1, b_2 \in \mathbb{N}$. In [MW2], when compared to [MW1], a simplification occurred in the definition of the parameters a_1, a_2, and Z. Here we come back to the definitions of these quantities given in [MW1]. The only changes are the following: we suppose $a_j \geqslant f l_j/d$ for $j = 1, 2$ and

$$1 \leqslant Z \leqslant \min\left\{\frac{dG}{\theta}, da_1, da_2, \text{Log}\left(\frac{2ea_1a_2d}{f(a_2l_1 + a_1l_2)}\right)\right\}.$$

Then there is nothing more to change in [MW2] except that, like in [MW1], we have to suppose $a_1b_1 \leqslant a_2b_2$.

In the present case $a := a_j = h(\alpha_j) = \frac{1}{3}\text{Log}(\lambda + 1)$, $j = 1, 2$ (so that we have $\sigma_1 = \sigma_2 = \sigma = 1$). We can take $f = 1$. We put $\tilde{B} = \max\{b_1, b_2, 4.4 \times 10^7\}$, and in this case $G = 1 + \text{Log}\, \tilde{B} + \text{Log Log}\, 2\tilde{B}$; thus $G > 21.5$.

Let

$$Z = \text{Log}\left(\frac{2ea_1a_2d}{f(a_2l_1 + a_1l_2)}\right) = \text{Log}\left(\frac{2el_1}{l_1 + l_2}\right) \cong \text{Log}(2e)$$

(thus $\varepsilon = 1$). Further we define $\theta = dG/Z$ (> 38) and $\theta_0 = \theta(Da/Z)$. The main term is $U = d^4a_1a_2G^2Z^{-3}$.

We apply Theorem 5.11 of [MW2]. If we choose $c_1 = 2.91$, $c = 16.6$, $c_0 = 403.94$, $\chi_0 = 1.056$, $\chi = 2.165$, then we get $|\Lambda| > e^{-\tilde{C}U}$ with

$$\tilde{C} = 3490.$$

Put $L = -\text{Log}\,|\Lambda|$ and $C' = \tilde{C}d^2Z^{-3}$ (thus $C' < 6473$). Then, by the estimates of $|\Lambda'|$ and q, the number L satisfies

$$\frac{D\,\text{Log}\,\lambda}{k_0} \leqslant L \leqslant C'(\text{Log}^2(\lambda + 1))G^2.$$

Since $\lambda\,\text{Log}\,\lambda \leqslant D$ and $n < \lambda$, we have

$$D \leqslant 6474(\text{Log}(\lambda + 1))G^2. \qquad (\#)$$

Suppose $n > 2.99 \times 10^6$. We have $D \geqslant n\,\text{Log}\, n > 4.45 \times 10^7$. But $D \geqslant C > E$ also, and so $D \geqslant \tilde{B}$. Thus $G \leqslant 1 + \text{Log}\, D + \text{Log Log}(2D)$, and

$$D < 6477\,\text{Log}(D/14.91)(1 + \text{Log}\, D + \text{Log Log}(2D))^2.$$

This implies $D < 4.474 \times 10^7$. Since $n\,\text{Log}\, n \leqslant D$, we obtain $n < 3 \times 10^6$.

3. The Numerical Verification

We already recalled that Thomas proved that $(*)$ has no non-trivial solution for $4 \leqslant n < 1000$. From now on we suppose $1000 < n \leqslant 3 \times 10^6$.

Consider the number

$$\kappa = \frac{\mathrm{Log}(\lambda)}{\mathrm{Log}(1 + 1/\lambda)}.$$

Since $\mathrm{Log}(1+x) = x - (1/2)x^2 + (1/3)x^3 - (\theta_1/4)x^4$, with $0 < \theta_1 < 1$ for $0 < x < 1$, we have

$$\lambda \,\mathrm{Log}(1 + \lambda^{-1}) = 1 - \frac{\lambda^{-1}}{2} + \frac{\lambda^{-2}}{3} - \frac{\theta_1 \lambda^{-3}}{4}.$$

This leads to

$$\frac{1}{\lambda \,\mathrm{Log}(1 + \lambda^{-1})} = 1 + \frac{\lambda^{-1}}{2} - \frac{\lambda^{-2}}{12} + \theta_2 \lambda^{-3}, \qquad |\theta_2| \leqslant 1/3.$$

So that

$$\kappa = \lambda \,\mathrm{Log}\,\lambda + \frac{1}{2}\,\mathrm{Log}\,\lambda - \frac{\mathrm{Log}\,\lambda}{12\lambda} + \frac{\theta_2 \,\mathrm{Log}\,\lambda}{\lambda^2}.$$

It is easy to verify that (for $n > 1000$)

$$\lambda = n + \frac{2}{n} - \frac{\theta_3}{n^2}, \qquad 0 < \theta_3 < 1.01.$$

If we substitute this estimate in the previous relation, we obtain

$$\kappa = n \,\mathrm{Log}\,\lambda + \frac{1}{2}\,\mathrm{Log}\,\lambda + \frac{23}{12n}\,\mathrm{Log}\,\lambda + \frac{\theta_4 \,\mathrm{Log}\,\lambda}{n^2}, \qquad |\theta_4| < 2.$$

Finally, notice that

$$\mathrm{Log}\,\lambda = \mathrm{Log}\,n + \mathrm{Log}\left(1 + \frac{2}{n^2} - \frac{\theta_3}{n^3}\right) = \mathrm{Log}\,n + \frac{2}{n^2} - \frac{\theta_5}{n^3}, \qquad 0 < \theta_5 < 1.02,$$

and thus

$$\kappa = n \,\mathrm{Log}\,n + \frac{1}{2}\,\mathrm{Log}\,n + \frac{23\,\mathrm{Log}\,n}{12n} + \frac{2}{n} + \frac{\theta \,\mathrm{Log}\,n}{n^2},$$

with $|\theta| < 2.5$ (note that $\theta \approx (-\theta_5 + 1)/\mathrm{Log}\,n + \theta_4$).

On the other hand, the estimate of Λ' in Section 1 implies

$$|E\kappa - C| < 1/(q^3 n^2) < e^{-n},$$

since $q^3 > \exp(n(\operatorname{Log} n)^2/k_0)$ and $k_0 < 2$ (see (19) in [T]).

So that, if there is a non-trivial solution for $n > 1000$, then

$$\left\| E\left(n \operatorname{Log} n + \frac{1}{2} \operatorname{Log} n + \frac{23 \operatorname{Log} n}{12n} + \frac{2}{n}\right) \right\| < \frac{3E \operatorname{Log} n}{n^2}.$$

(For x real, we put $\|x\| = x - [x + 0.5] = \operatorname{dist}(x, \mathbb{Z})$.)

Recall that $D \geqslant E\lambda \operatorname{Log} \lambda$; thus

$$E \leqslant \frac{D}{n \operatorname{Log} n}.$$

For example, $E \leqslant 1$ for $n \geqslant 1.6 \times 10^6$.

Because of these facts, we work in two different manners for $n > 5 \times 10^5$ and $n \leqslant 5 \times 10^5$.

In the first range, we have $E \leqslant 7$ and we have only to verify that

$$\left\| E\left(n \operatorname{Log} n + \frac{1}{2} \operatorname{Log} n + \frac{23 \operatorname{Log} n}{12n} + \frac{2}{n}\right) \right\| > \frac{21 \operatorname{Log} n}{n^2}$$

for $5 \times 10^5 < n < 3 \times 10^6$.

Using language C, this first verification took less than one hour on an Apollo 3500 computer.

On the range $10^3 < n \leqslant 5 \times 10^5$, we compute the continued fraction expansion of the number κ until we reach the bound $\tilde{E}(n) := 4.48 \times 10^7/(n \operatorname{Log} n)$ for the denominator. Let $10^3 < n \leqslant 5 \times 10^6$. If there is a principal convergent, p/q, for κ_n such that

$$\tilde{E}(n) < q < e^{-n/2}$$

then $(*)$ has no non-trivial solution for that value of n. And we verify that $\|E\kappa_n\| > e^{-100}$ for $10^3 < n \leqslant 5 \times 10^5$ and $1 \leqslant E \leqslant \tilde{E}(n)$. This second verification took about five hours.

This shows that $(*)$ has no non-trivial solution in the interval $10^3 < n \leqslant 3 \times 10^6$.

This verification was made with some care. In the range $n \geqslant 5 \times 10^5$, when $\|E\kappa\|$ is small (indeed, for $\|E\kappa\| < 10^{-5}$), an independent verification was realized by BC, the "super-calculator" given with the UNIX system. Notice that BC enables one to compute with 80 exact digits. In the range $10^3 < n < 5 \times 10^5$, when the continued fraction of κ has a large coefficient (i.e., $> 10^3$), then the computation was verified with BC and also with a special program which gives about 150 exact digits.

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